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Quantum mechanics on a sphere and coherent states

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Abstract. The coherent states for a particle on a sphere are introduced. These states are labelled by points of the classical phase space, i.e. the position on the sphere and the angular momentum of a particle. As with the coherent states for a particle on a circle discussed in Kowalski *et al* (1996 *J. Phys. A: Math. Gen.* **29** 4149), we deal with a deformation of the classical phase space related to quantum fluctuations. The expectation values of the position and the angular momentum in the coherent states are regarded as the best possible approximation of the classical phase space. The correctness of the introduced coherent states is illustrated by an example of the rotator.

1. Introduction

It has become a cliché to say that coherent states abound in quantum physics [1]. Moreover, it turns out that they can also be applied in the theory of quantum deformations [2] and even in the theory of classical dynamical systems [3].

In spite of the fact that the problem of quantization of particle motion on a sphere is at least 70 years old, there still remains an open question concerning the coherent states for a particle on a sphere. Indeed, the celebrated spin coherent states introduced by Radcliffe [4] and Perelomov [5] are labelled by points of a sphere, i.e. the elements of the configuration space. On the other hand, it seems that as with the standard coherent states, the coherent states for a particle on a sphere should be marked with points of the phase space rather than the configuration space.

The aim of this work is to introduce the coherent states for a quantum particle on the sphere S^2, labelled by points of the phase space, that is the cotangent bundle T^*S^2. The construction follows the general scheme introduced in [6] for the case of motion in a circle, based on the polar decomposition of the operator defining via the eigenvalue equation the coherent states. From the technical point of view our treatment utilizes both the Barut–Girardello [7] and Perelomov approach [5]. Namely, as with the Barut–Girardello approach the coherent states are defined as the eigenvectors of some non-Hermitian operators. On the other hand, in analogy to the Perelomov formalism those states are generated from some 'vacuum vector', nevertheless in opposition to the Perelomov group-theoretic construction, the coherent states are obtained by means of the non-unitary action.

In section 2 we recall the construction of the coherent states for a particle on a circle. Sections 3–6 are devoted to the definition of the coherent states for a particle on a sphere and discussion of their most important properties. For an easy illustration of the introduced approach we study in section 7 the case with free motion on a sphere.

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2. Coherent states for a particle on a circle

In this section we recall the basic properties of the coherent states for a particle on a circle introduced in [6]. Consider the case of free motion in a circle. For the sake of simplicity we assume that the particle has unit mass and it moves in a unit circle. The classical Lagrangian is

$$L = \tfrac{1}{2}\dot{\varphi}^2 \tag{2.1}$$

so the angular momentum canonically conjugate to the angle φ is given by

$$J = \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} \tag{2.2}$$

and the Hamiltonian can be written as

$$H = \tfrac{1}{2}J^2. \tag{2.3}$$

Evidently, we have the Poisson bracket of the form

$$\{\varphi, J\} = 1 \tag{2.4}$$

implying according to the rules of the canonical quantization the commutator

$$[\hat{\varphi}, \hat{J}] = \mathrm{i} \tag{2.5}$$

where we set $\hbar = 1$. The operator $\hat{\varphi}$ does not take into account the topology of the circle and (2.5) needs very subtle analysis. The better candidate to represent the position of a particle on the unit circle is the unitary operator U

$$U = \mathrm{e}^{\mathrm{i}\hat{\varphi}}. \tag{2.6}$$

Indeed, the substitution $\hat{\varphi} \to \hat{\varphi} + 2n\pi$ does not change U, i.e. U preserves the topology of the circle. The operator U leads to the algebra

$$[\hat{J}, U] = U \tag{2.7}$$

where U is unitary. We would like to stress that (2.6) was used only for the heuristic derivation of (2.7) which is the correct quantization rule. Consider the eigenvalue equation

$$\hat{J}|j\rangle = j|j\rangle. \tag{2.8}$$

Using (2.7) and (2.8) we find that the operators U and $U^\dagger$ are the ladder operators, namely

$$U|j\rangle = |j+1\rangle \tag{2.9a}$$

$$U^\dagger|j\rangle = |j-1\rangle. \tag{2.9b}$$

Demanding the time-reversal invariance of representations of the algebra (2.7), we conclude [6] that the eigenvalues j of the operator $\hat{J}$ can be only integer (boson case) or half-integer (fermion case).

Now consider the eigenvalue equation satisfied by the standard coherent states $|z\rangle$ [8, 9] with complex z, of the form

$$\mathrm{e}^{\mathrm{i}\hat{a}}|z\rangle = \mathrm{e}^{\mathrm{i}z}|z\rangle \tag{2.10}$$

where $\hat{a} \sim \hat{q} + \mathrm{i}\hat{p}$ is the standard Bose annihilation operator and $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively. In analogy with (2.10) we define the coherent states $|\xi\rangle$ for a particle on a circle, where ξ is a complex coordinate parametrizing the classical phase space, by means of the eigenvalue equation

$$Z|\xi\rangle = \xi|\xi\rangle \tag{2.11}$$

where

$$Z := \mathrm{e}^{\mathrm{i}(\hat{\varphi}+\mathrm{i}\hat{J})}. \tag{2.12}$$

Hence, making use of the Baker–Hausdorff formula we obtain

$$Z = \mathrm{e}^{-\hat{J}+\frac{1}{2}}U. \tag{2.13}$$

We remark that the complex number ξ should parametrize the cylinder which is the classical phase space for the particle moving in a circle. The convenient parametrization of ξ consistent with the form of the operator Z such that

$$\xi = \mathrm{e}^{-l+\mathrm{i}\varphi} \tag{2.14}$$

arises from the deformation of the circular cylinder by means of the transformation

$$x = \mathrm{e}^{-l}\cos\varphi \qquad y = \mathrm{e}^{-l}\sin\varphi \qquad z = l. \tag{2.15}$$

The coherent states $|\xi\rangle$ can be represented as

$$|\xi\rangle = \mathrm{e}^{-(\ln\xi)\hat{J}}|1\rangle \tag{2.16}$$

where

$$|1\rangle = \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}j^2}|j\rangle. \tag{2.17}$$

The coherent states satisfy

$$\frac{\langle\xi|\hat{J}|\xi\rangle}{\langle\xi|\xi\rangle} \approx l \tag{2.18}$$

where the maximal error arising in the case $l \to 0$ is of the order of 0.1% and we have the exact equality in the case with l integer or half-integer. Therefore, l can be identified with the classical angular momentum. Furthermore, we have

$$\frac{\langle\xi|U|\xi\rangle}{\langle\xi|\xi\rangle} \approx \mathrm{e}^{-1/4}\mathrm{e}^{\mathrm{i}\varphi}. \tag{2.19}$$

It thus appears that the average value of U in the normalized coherent state does not belong to the unit circle. On introducing the relative average of U of the form

$$\frac{\langle U\rangle_\xi}{\langle U\rangle_\eta} := \frac{\langle\xi|U|\xi\rangle}{\langle\eta|U|\eta\rangle} \tag{2.20}$$

where $|\xi\rangle$ and $|\eta\rangle$ are the normalized coherent states, we find

$$\frac{\langle U\rangle_\xi}{\langle U\rangle_1} \approx \mathrm{e}^{\mathrm{i}\varphi}. \tag{2.21}$$

From (2.21) it follows that the relative expectation value $\langle U\rangle_\xi/\langle U\rangle_1$ is the most natural candidate to describe the average position of a particle on a circle and φ can be regarded as the classical angle.

We remark that the coherent states on the circle have been recently discussed by Gonzáles *et al* [10]. In spite of the fact that they formally generalize the coherent states described above, the ambiguity of the definition of those states manifesting in their dependence on some extra parameter, can be avoided only by demanding the time-reversal invariance mentioned earlier, which leads precisely to the coherent states introduced in [6]. Since the time-reversal symmetry seems to be fundamental one for the motion of the classical particle in a circle and makes the quantization unique, therefore the generalization of the coherent states discussed in

[10] which does not preserve that symmetry is of interest rather from the mathematical point of view.

Bearing in mind the properties of the standard coherent states, one may ask about the minimalization of the Heisenberg uncertainty relations by the introduced coherent states for a particle on a circle. In our opinion, in the case with the compact manifolds the minimalization of the Heisenberg uncertainty relations is not an adequate tool for the definition of the coherent states. A counterexample can be easily deduced from (2.7)–(2.9). Indeed, taking into account (2.8) and (2.9) we find that for the eigenvectors $|j\rangle$s of the angular momentum $\hat{J}$ the equality sign is attended in the Heisenberg uncertainty relations implied by (2.7) such that

$$(\Delta\hat{J})^2 \geqslant \frac{1}{4}\frac{|\langle U\rangle|^2}{1-|\langle U\rangle|^2}. \tag{2.22}$$

More precisely, for these states (2.22) takes the form $0 = 0$. On the other hand, the vectors $|j\rangle$s are clearly a rather poor candidate for the coherent states. In our opinion the fact that the coherent states are 'the most classical' ones is better described by the following easily proven formulae:

$$(\Delta\hat{J})^2 \approx \text{constant} \tag{2.23}$$

$$\frac{\langle U^2\rangle}{\langle U\rangle^2} \approx \text{constant} \tag{2.24}$$

where the approximations are very good ones. In fact, these relations mean that the quantum variables $\hat{J}$ and U are at a practically constant 'distance' from their classical counterparts $\langle\hat{J}\rangle$ and $\langle U\rangle$, respectively, and therefore the quantum observables and the corresponding expectation values connected with the classical phase space are mutually related. We point out that in the case with the standard coherent states for a particle on a real line we have the exact formulae

$$(\Delta\hat{p})^2 = \text{constant} \tag{2.25}$$

$$(\Delta\hat{q})^2 = \text{constant}. \tag{2.26}$$

It seems to us that the approximative nature of the relations (2.23) and (2.24) is related to the compactness of the circle.

3. Unitary representations of the $e(3)$ algebra and quantum mechanics on a sphere

Our experience with the case of the circle discussed in the previous section indicates that in order to introduce the coherent states we should first identify the algebra adequate for the study of the motion on a sphere. The fact that the algebra (2.7) referring to the case with the circle S^1 is equivalent to the $e(2)$ algebra, where $E(2)$ is the group of the plane consisting of translations and rotations,

$$[\hat{J}, X_\alpha] = \mathrm{i}\varepsilon_{\alpha\beta}X_\beta \qquad [X_\alpha, X_\beta] = 0 \qquad \alpha, \beta = 1, 2 \tag{3.1}$$

realized in a unitary irreducible representation by Hermitian operators

$$X_1 = r(U + U^\dagger)/2 \qquad X_2 = r(U - U^\dagger)/2\mathrm{i} \tag{3.2}$$

where the Casimir is

$$X_1^2 + X_2^2 = r^2 \tag{3.3}$$

and $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor, indicates that the most natural algebra for the case with the sphere S^2 is the $e(3)$ algebra such that

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i, X_j] = \mathrm{i}\varepsilon_{ijk}X_k \qquad [X_i, X_j] = 0 \qquad i, j, k = 1, 2, 3. \tag{3.4}$$

Indeed, the algebra (3.4) has two Casimir operators given in a unitary irreducible representation by

$$\boldsymbol{X}^2 = r^2 \qquad \boldsymbol{J}\cdot\boldsymbol{X} = \lambda \tag{3.5}$$

where the dot designates the scalar product. Therefore, as with the generators X_α, $\alpha = 1, 2$, describing the position of a particle on the circle, the generators X_i, $i = 1, 2, 3$, can be regarded as quantum counterparts of the Cartesian coordinates of the points of the sphere S^2 with radius r. We point out that unitary irreducible representations of (3.4) can be labelled by r and the new scale-invariant parameter $\zeta = \frac{\lambda}{r}$. It is clear that ζ is simply the projection of the angular momentum $\boldsymbol{J}$ on the direction of the radius vector of a particle. Since we did not find any denomination for such an entity in the literature, therefore we have decided to call ζ the *twist* of a particle.

Let us now recall the basic properties of the unitary representations of the $e(3)$ algebra. The $e(3)$ algebra expressed with the help of operators J_3, $J_\pm = J_1 \pm \mathrm{i}J_2$, X_3 and $X_\pm = X_1 \pm \mathrm{i}X_2$, takes the form

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm \tag{3.6a}$$

$$[J_\pm, X_\mp] = \pm 2X_3 \qquad [J_\pm, X_\pm] = 0 \qquad [J_\pm, X_3] = \mp X_\pm \tag{3.6b}$$

$$[J_3, X_\pm] = \pm X_\pm \qquad [J_3, X_3] = 0 \tag{3.6c}$$

$$[X_+, X_-] = [X_\pm, X_3] = 0. \tag{3.6d}$$

Consider the irreducible representation of the above algebra in the angular momentum basis spanned by the common eigenvectors $|j, m; r, \zeta\rangle$ of the operators $\boldsymbol{J}^2 = J_+J_- + J_3^2 - J_3$, J_3, $\boldsymbol{X}^2$ and $\boldsymbol{J}\cdot\boldsymbol{X}/r$

$$\boldsymbol{J}^2|j, m; r, \zeta\rangle = j(j+1)|j, m; r, \zeta\rangle \qquad J_3|j, m; r, \zeta\rangle = m|j, m; r, \zeta\rangle \tag{3.7a}$$

$$\boldsymbol{X}^2|j, m; r, \zeta\rangle = r^2|j, m; r, \zeta\rangle \qquad (\boldsymbol{J}\cdot\boldsymbol{X}/r)|j, m; r, \zeta\rangle = \zeta|j, m; r, \zeta\rangle \tag{3.7b}$$

where $-j \leqslant m \leqslant j$. Recall that

$$J_\pm|j, m; r, \zeta\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1; r, \zeta\rangle. \tag{3.8}$$

The operators $X_\pm$ and X_3 act on the vectors $|j, m; r, \zeta\rangle$ in the following way:

$$\begin{aligned} X_+|j, m; r, \zeta\rangle = {} & -\frac{r\sqrt{(j+1)^2-\zeta^2}\sqrt{(j+m+1)(j+m+2)}}{(j+1)\sqrt{(2j+1)(2j+3)}}|j+1, m+1; r, \zeta\rangle \\ & + \frac{\zeta r\sqrt{(j-m)(j+m+1)}}{j(j+1)}|j, m+1; r, \zeta\rangle \\ & + \frac{r\sqrt{j^2-\zeta^2}\sqrt{(j-m-1)(j-m)}}{j\sqrt{(2j-1)(2j+1)}}|j-1, m+1; r, \zeta\rangle \end{aligned} \tag{3.9a}$$

$$\begin{aligned} X_-|j, m; r, \zeta\rangle = {} & \frac{r\sqrt{(j+1)^2-\zeta^2}\sqrt{(j-m+1)(j-m+2)}}{(j+1)\sqrt{(2j+1)(2j+3)}}|j+1, m-1; r, \zeta\rangle \\ & + \frac{\zeta r\sqrt{(j-m+1)(j+m)}}{j(j+1)}|j, m-1; r, \zeta\rangle \\ & - \frac{r\sqrt{j^2-\zeta^2}\sqrt{(j+m-1)(j+m)}}{j\sqrt{(2j-1)(2j+1)}}|j-1, m-1; r, \zeta\rangle \end{aligned} \tag{3.9b}$$

$$X_3|j,m;r,\zeta\rangle = \frac{r\sqrt{(j+1)^2-\zeta^2}\sqrt{(j-m+1)(j+m+1)}}{(j+1)\sqrt{(2j+1)(2j+3)}}|j+1,m;r,\zeta\rangle$$
$$+\frac{\zeta rm}{j(j+1)}|j,m;r,\zeta\rangle + \frac{r\sqrt{j^2-\zeta^2}\sqrt{(j-m)(j+m)}}{j\sqrt{(2j-1)(2j+1)}}|j-1,m;r,\zeta\rangle. \quad (3.9c)$$

An immediate consequence of (3.9) is the existence of the minimal $j = j_{\min}$ satisfying

$$j_{\min} = |\zeta|. \quad (3.10)$$

Thus, it turns out that in the representation defined by (3.9) the twist ζ can be only integer or half-integer. We finally write down the orthogonality and completeness conditions satisfied by the vectors $|j,m;r,\zeta\rangle$ such that

$$\langle j,m;r,\zeta|j',m';r,\zeta\rangle = \delta_{jj'}\delta_{mm'} \quad (3.11)$$

$$\sum_{j=|\zeta|}^{\infty}\sum_{m=-j}^{j}|j,m;r,\zeta\rangle\langle j,m;r,\zeta| = I \quad (3.12)$$

where I is the identity operator.

4. Definition of coherent states for a particle on a sphere

Now, our experience with the circle indicates that one should identify by means of the $e(3)$ algebra an analogue of the unitary operator U (2.6), representing the position of a particle on a sphere. To do this, let us recall that a counterpart of the 'position' $\mathrm{e}^{\mathrm{i}\varphi}$ on the circle S^1 is a unit length imaginary quaternion which can be represented with the help of the Pauli matrices σ_i, $i = 1, 2, 3$, as

$$\eta = \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma} \quad (4.1)$$

where $\boldsymbol{n}^2 = 1$. Note that η is simply an element of the $SU(2)$ group and it is related to the $S^2 \approx SU(2)/U(1)$ quotient space. Therefore, the most natural choice for the 'position operator' of a particle on a sphere is to set

$$V = \frac{1}{r}\boldsymbol{\sigma}\cdot\boldsymbol{X} \quad (4.2)$$

where X_i, $i = 1, 2, 3$ obey (3.4) and (3.9) and we have omitted for convenience the imaginary factor i. Furthermore, let us introduce a version of the Dirac matrix operator [11]

$$K := -(\boldsymbol{\sigma}\cdot\boldsymbol{J} + 1). \quad (4.3)$$

Observe that

$$V^\dagger = V \qquad K^\dagger = K. \quad (4.4)$$

Making use of the operators V and K we can write the relations defining the $e(3)$ algebra in the space of the unitary irreducible representation introduced above as

$$(\mathrm{Tr}\,\boldsymbol{\sigma}K)^2 = 4K(K+1) \quad (4.5a)$$

$$[K, V]_+ = \mathrm{Tr}\,KV \quad (4.5b)$$

$$V^2 = I \quad (4.5c)$$

where $\mathrm{Tr}\,A = A_{11} + A_{22}$ and the subscript '+' designates the anticommutator. In particular,

$$\mathrm{Tr}\,KV = -2\boldsymbol{J}\cdot\boldsymbol{X}/r = -2\zeta. \quad (4.6)$$

It should also be noted that in view of (4.4) and (4.5*c*) V satisfies the unitarity condition $V^{\dagger}V = I$.

We now introduce the vector operator $\boldsymbol{Z}$ generating, via the eigenvalue equation analogous to (2.11), the coherent states for a particle on a sphere S^2. The experience with the circle (see equation (2.13)) suggests the following form of the 'polar decomposition' for the matrix operator counterpart Z of the operator $\boldsymbol{Z}$:

$$Z = \mathrm{e}^{-K} V. \tag{4.7}$$

Indeed, it is easy to see that in the case of the circular motion in the equator defined semiclassically by $J_1 = J_2 = 0$ and $X_3 = 0$, Z reduces to the diagonal matrix operator with Z given by (2.13) and its Hermitian conjugate on the diagonal. Furthermore, using (4.5*b*) we find

$$Z - Z^{-1} = 2\zeta K^{-1} \sinh K. \tag{4.8}$$

Motivated by the complexity of the problem we now restrict to the simplest case of the twist $\zeta = 0$ when (4.8) takes the form

$$Z^2 = I. \tag{4.9}$$

In the following we confine ourselves to the case $\zeta = 0$. The general case with arbitrary $\zeta \neq 0$ will be discussed in a separate work. Besides (4.9) we also have the remarkably simple relation (4.5*b*) referring to $\zeta = 0$ such that

$$[K, V]_+ = 0. \tag{4.10}$$

Note that the case $\zeta = 0$ is the 'most classical' one. Indeed, the projection of the angular momentum onto the direction of the radius vector should vanish for the classical particle on a sphere. It should also be noted that in view of (3.10) js and ms labelling the basis vectors $|j, m; r, \zeta\rangle$ are integer in the case of the twist $\zeta = 0$. We finally point out that the condition $\zeta = 0$ ensures the invariance of the irreducible representation of the $e(3)$ algebra under time inversions and parity transformations which change the sign of the product $\boldsymbol{J} \cdot \boldsymbol{X}$. Clearly, demanding the time reversal or the parity invariance when $\zeta \neq 0$ one should work with representations involving both ζ and $-\zeta$.

We now return to (4.7). Making use of (4.10) and the fact that the matrix operator V in view of (4.2) is a traceless one we obtain for $\zeta = 0$

$$\mathrm{Tr}\, Z = 0. \tag{4.11}$$

Hence,

$$Z = \boldsymbol{\sigma} \cdot \boldsymbol{Z}. \tag{4.12}$$

Taking into account (4.9) we obtain from (4.12)

$$\boldsymbol{Z}^2 = 1 \tag{4.13}$$

and

$$[Z_i, Z_j] = 0 \qquad i, j = 1, 2, 3. \tag{4.14}$$

As with (4.2) describing in the matrix language the position of a quantum particle on a sphere, the matrix operator (4.12) can be only interpreted as a convenient arrangement of the operators Z_i generating the coherent states, simplifying the algebraic analysis of the problem. Accordingly, we define the coherent states for a quantum mechanics on a sphere in terms of operators Z_i, as the solutions of the eigenvalue equation such that

$$\boldsymbol{Z}|\boldsymbol{z}\rangle = \boldsymbol{z}|\boldsymbol{z}\rangle \tag{4.15}$$

where in view of (4.13) $\boldsymbol{z}^2 = 1$. What is $\boldsymbol{Z}$? Using (4.7), (4.2), (4.3) and setting $\zeta = 0$, we find after some calculation

$$\boldsymbol{Z} = \left(\frac{\mathrm{e}^{1/2}}{\sqrt{1+4\boldsymbol{J}^2}}\sinh\tfrac{1}{2}\sqrt{1+4\boldsymbol{J}^2} + \mathrm{e}^{1/2}\cosh\tfrac{1}{2}\sqrt{1+4\boldsymbol{J}^2}\right)\frac{\boldsymbol{X}}{r} + \mathrm{i}\left(\frac{2\mathrm{e}^{1/2}}{\sqrt{1+4\boldsymbol{J}^2}}\sinh\tfrac{1}{2}\sqrt{1+4\boldsymbol{J}^2}\right)\boldsymbol{J}\times\frac{\boldsymbol{X}}{r}. \tag{4.16}$$

We remark that Z_i have the structure resembling the standard annihilation operators. In fact, one can easily check that it can be written as a combination

$$\boldsymbol{Z} = a\boldsymbol{X} + \mathrm{i}b\boldsymbol{P} \tag{4.17}$$

of the 'position operator' $\boldsymbol{X}$ and the 'momentum' $\boldsymbol{P}$, where the coefficients a and b are functions of $\boldsymbol{J}^2$. We finally point out that derivation of the operator $\boldsymbol{Z}$ (4.16) without knowledge of the matrix operator Z seems to be a very difficult task.

5. Construction of the coherent states

In this section we construct the coherent states specified by the eigenvalue equation (4.15). On projecting (4.15) on the basis vectors $|j,m;r\rangle \equiv |j,m;r,0\rangle$ and using (3.7a), (3.8) and (3.9) with $\zeta = 0$ we arrive at the system of linear difference equations satisfied by the Fourier coefficients of the expansion of the coherent state $|\boldsymbol{z}\rangle$ in the basis $|j,m;r\rangle$. The direct solution of such a system in the general case seems to be a difficult task. Therefore, we adopt the following technique. We first solve the eigenvalue equation for $\boldsymbol{z} = \boldsymbol{n}_3 = (0,0,1)$, and then generate the coherent states from the vector $\boldsymbol{n}_3$ using the fact (see (4.16)) that $\boldsymbol{Z}$ is a vector operator. As demonstrated in the next section the case with $\boldsymbol{z} = \boldsymbol{n}_3$ refers to $\boldsymbol{x} = (0,0,1)$ and $\boldsymbol{l} = \boldsymbol{0}$, where $\boldsymbol{x}$ is the position and $\boldsymbol{l}$ the angular momentum, respectively, i.e. the particle resting on the 'north pole' of the sphere. Let us write down the eigenvalue equation (4.15) for $\boldsymbol{z} = \boldsymbol{n}_3$,

$$\boldsymbol{Z}|\boldsymbol{n}_3\rangle = \boldsymbol{n}_3|\boldsymbol{n}_3\rangle. \tag{5.1}$$

Using the following relations which can be easily derived with the help of (4.16), (3.7a), (3.8) and (3.9) with $\zeta = 0$:

$$\begin{aligned} Z_1|j,m;r\rangle = &-\frac{1}{2}\mathrm{e}^{-j-1}\sqrt{\frac{(j+m+1)(j+m+2)}{(2j+1)(2j+3)}}|j+1,m+1;r\rangle \\ &+\frac{1}{2}\mathrm{e}^{j}\sqrt{\frac{(j-m-1)(j-m)}{(2j-1)(2j+1)}}|j-1,m+1;r\rangle \\ &+\frac{1}{2}\mathrm{e}^{-j-1}\sqrt{\frac{(j-m+1)(j-m+2)}{(2j+1)(2j+3)}}|j+1,m-1;r\rangle \\ &-\frac{1}{2}\mathrm{e}^{j}\sqrt{\frac{(j+m-1)(j+m)}{(2j-1)(2j+1)}}|j-1,m-1;r\rangle \end{aligned} \tag{5.2a}$$

$$
\begin{aligned}
Z_2|j,m;r\rangle = {}& \frac{\mathrm{i}}{2}\mathrm{e}^{-j-1}\sqrt{\frac{(j+m+1)(j+m+2)}{(2j+1)(2j+3)}}|j+1,m+1;r\rangle \\
& - \frac{\mathrm{i}}{2}\mathrm{e}^{j}\sqrt{\frac{(j-m-1)(j-m)}{(2j-1)(2j+1)}}|j-1,m+1;r\rangle \\
& + \frac{\mathrm{i}}{2}\mathrm{e}^{-j-1}\sqrt{\frac{(j-m+1)(j-m+2)}{(2j+1)(2j+3)}}|j+1,m-1;r\rangle \\
& - \frac{\mathrm{i}}{2}\mathrm{e}^{j}\sqrt{\frac{(j+m-1)(j+m)}{(2j-1)(2j+1)}}|j-1,m-1;r\rangle
\end{aligned} \tag{5.2b}
$$

$$
\begin{aligned}
Z_3|j,m;r\rangle = {}& \mathrm{e}^{-j-1}\sqrt{\frac{(j-m+1)(j+m+1)}{(2j+1)(2j+3)}}|j+1,m;r\rangle \\
& + \mathrm{e}^{j}\sqrt{\frac{(j-m)(j+m)}{(2j-1)(2j+1)}}|j-1,m;r\rangle
\end{aligned} \tag{5.2c}
$$

it can be easily checked that the solution to (5.1) is given by

$$
|\boldsymbol{n}_3\rangle = \sum_{j=0}^{\infty} \mathrm{e}^{-\frac{1}{2}j(j+1)}\sqrt{2j+1}|j,0;r\rangle. \tag{5.3}
$$

Now, using the commutator

$$
[\boldsymbol{w}\cdot\boldsymbol{J}, \boldsymbol{Z}] = -\mathrm{i}\boldsymbol{w}\times\boldsymbol{Z} \tag{5.4}
$$

where $\boldsymbol{w} \in \mathbb{C}^3$, we generate the complex rotation of $\boldsymbol{Z}$

$$
\mathrm{e}^{\boldsymbol{w}\cdot\boldsymbol{J}}\boldsymbol{Z}\mathrm{e}^{-\boldsymbol{w}\cdot\boldsymbol{J}} = \cosh\sqrt{\boldsymbol{w}^2}\,\boldsymbol{Z} - \mathrm{i}\frac{\sinh\sqrt{\boldsymbol{w}^2}}{\sqrt{\boldsymbol{w}^2}}\boldsymbol{w}\times\boldsymbol{Z} + \frac{1-\cosh\sqrt{\boldsymbol{w}^2}}{\boldsymbol{w}^2}\boldsymbol{w}(\boldsymbol{w}\cdot\boldsymbol{Z}). \tag{5.5}
$$

Taking into account (5.5) and (4.15) we find that the coherent states can be expressed by

$$
|\boldsymbol{z}\rangle = \mathrm{e}^{\boldsymbol{w}\cdot\boldsymbol{J}}|\boldsymbol{n}_3\rangle \tag{5.6}
$$

where $\boldsymbol{w}$ is given by

$$
\boldsymbol{w} = \frac{\operatorname{arccosh} z_3}{\sqrt{1-z_3^2}}\boldsymbol{z}\times\boldsymbol{n}_3. \tag{5.7}
$$

It thus appears that the coherent states can be written as

$$
|\boldsymbol{z}\rangle = \exp\left[\frac{\operatorname{arccosh} z_3}{\sqrt{1-z_3^2}}(\boldsymbol{z}\times\boldsymbol{n}_3)\cdot\boldsymbol{J}\right]|\boldsymbol{n}_3\rangle. \tag{5.8}
$$

We remark that the discussed coherent states are generated analogously as in the case of the circle described by equation (2.16). The formula (5.8) can be furthermore written in the form

$$
|\boldsymbol{z}\rangle = \mathrm{e}^{\mu J_-}\mathrm{e}^{\gamma J_3}\mathrm{e}^{\nu J_+}|\boldsymbol{n}_3\rangle \tag{5.9}
$$

where

$$
\mu = \frac{z_1+\mathrm{i}z_2}{1+z_3} \qquad \nu = \frac{-z_1+\mathrm{i}z_2}{1+z_3} \qquad \gamma = \ln\frac{1+z_3}{2}. \tag{5.10}
$$

Finally, equations (5.9), (5.3), (3.7a) and (3.8) taken together yield the following formula on the coherent states:

$$|z\rangle = \sum_{j=0}^{\infty} \mathrm{e}^{-\frac{1}{2}j(j+1)}\sqrt{2j+1}\sum_{m=0}^{j}\frac{\nu^m}{m!}\frac{(j+m)!}{(j-m)!}\mathrm{e}^{\gamma m}\sum_{k=0}^{j+m}\frac{\mu^k}{k!}\sqrt{\frac{(j-m+k)!}{(j+m-k)!}}|j, m-k; r\rangle \quad (5.11)$$

where μ, ν and γ are expressed by (5.10) and $\boldsymbol{z}^2 = 1$. Taking into account the identities

$$\sum_{s=0}^{n}\frac{(s+k)!}{(s+m)!s!(n-s)!}z^s = \frac{k!}{m!n!}\,{}_2F_1(-n, k+1, m+1; -z) \quad (5.12)$$

$$C_n^{\alpha}(x) = \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)}\,{}_2F_1(-n, n+2\alpha, \alpha+\tfrac{1}{2}; \tfrac{1}{2}(1-x)) \quad (5.13)$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function, $C_n^{\alpha}(x)$ are the Gegenbauer polynomials and $\Gamma(x)$ is the gamma function, we obtain

$$\langle j, m; r|z\rangle = \mathrm{e}^{-\frac{1}{2}j(j+1)}\sqrt{2j+1}\,\frac{(2|m|)!}{|m|!}\sqrt{\frac{(j-|m|)!}{(j+|m|)!}}\left(\frac{-\varepsilon(m)z_1 + \mathrm{i}z_2}{2}\right)^{|m|}C_{j-|m|}^{|m|+\frac{1}{2}}(z_3) \quad (5.14)$$

where $\varepsilon(m)$ is the sign of m. Let us recall in the context of the relations (5.14) that the polynomial dependence of the projection of coherent states onto the discrete basis vectors, on the complex numbers parametrizing those states is one of their most characteristic properties. Clearly, the polynomials (5.14) should span via the 'resolution of the identity operator' the Fock–Bargmann representation. We recall that the existence of such a representation is one of the most important properties of coherent states. The problem of finding the Fock–Bargmann representation in the discussed case of the coherent states for a particle on a sphere is technically complicated and it will be discussed in a separate work. Finally, note that the coherent states $|z\rangle$ are evidently stable under rotations.

6. Coherent states and the classical phase space

We now show that the introduced coherent states for a quantum particle on a sphere are labelled by points of the classical phase space, that is T^*S^2. Referring back to equation (4.16) and taking into account the fact that the classical limit corresponds to large js, we arrive at the following parametrization of $\boldsymbol{z}$ by points of the phase space:

$$\boldsymbol{z} = \cosh|\boldsymbol{l}|\,\frac{\boldsymbol{x}}{r} + \mathrm{i}\frac{\sinh|\boldsymbol{l}|}{|\boldsymbol{l}|}\,\boldsymbol{l}\times\frac{\boldsymbol{x}}{r} \quad (6.1)$$

where the vectors $\boldsymbol{l}, \boldsymbol{x} \in \mathbb{R}^3$, fulfil $\boldsymbol{x}^2 = r^2$ and $\boldsymbol{l}\cdot\boldsymbol{x} = 0$, i.e. we assume that $\boldsymbol{l}$ is the classical angular momentum and $\boldsymbol{x}$ is the radius vector of a particle on a sphere. In accordance with the formulae (4.15) and (4.13) the vector $\boldsymbol{z}$ satisfies $\boldsymbol{z}^2 = 1$. Thus, the vector $\boldsymbol{z}$ is really parametrized by the points $(\boldsymbol{x}, \boldsymbol{l})$ of the classical phase space T^*S^2.

Consider now the expectation value of the angular momentum operator $\boldsymbol{J}$ in a coherent state. The explicit formulae which can be derived with the help of (3.7a), (3.8), (3.12) and (5.14) are too complicated to reproduce them herein. From computer simulations it follows that

$$\langle \boldsymbol{J}\rangle_{\boldsymbol{z}} = \frac{\langle \boldsymbol{z}|\boldsymbol{J}|\boldsymbol{z}\rangle}{\langle \boldsymbol{z}|\boldsymbol{z}\rangle} \approx \boldsymbol{l}. \quad (6.2)$$

Nevertheless, in opposition to the case with circular motion, the approximate relation (6.2) does not hold for practically arbitrarily small $|\boldsymbol{l}|$. Namely, we have found that whenever $|\boldsymbol{l}| \sim 1$, then (6.2) is not valid. Note that returning to dimensional entities in the formulae like (3.6) we measure $|\boldsymbol{l}|$ in units of $\hbar$, so in physical units we deal rather with $\boldsymbol{L} = \hbar\boldsymbol{l}$. For $|\boldsymbol{l}| \geqslant 10$ the relative error $|(\langle J_i\rangle_z - l_i)/\langle J_i\rangle_z|$, $i = 1, 2, 3$, is small. More precisely, if $|\boldsymbol{l}| \sim 10$, then $|(\langle J_i\rangle_z - l_i)/\langle J_i\rangle_z| \sim 1\%$. In other words, in the case of motion on a sphere, the quantum fluctuations are not negligible for $|\boldsymbol{L}| \sim 1\hbar$ and the description based on the concept of the classical phase space is not an adequate one. However, it must be borne in mind that the condition $|\boldsymbol{L}| \geqslant 10\,\hbar$, when (6.2) holds is not the same as the classical limit $|\boldsymbol{l}| \to \infty$. We only point out that $10\,\hbar \approx 10^{-33}$ J s. It thus appears that the parameter $\boldsymbol{l}$ in (6.2) can be identified with the classical angular momentum divided by $\hbar$.

We now study the role of the parameter $\boldsymbol{x}$ in (6.1). As with the momentum operator $\boldsymbol{J}$ the explicit relations obtained by means of (3.9) with $\zeta = 0$, equations (3.12) and (5.14) are too complicated to write them down herein. The computer simulations indicate that

$$\langle \boldsymbol{X}\rangle_z = \frac{\langle z|\boldsymbol{X}|z\rangle}{\langle z|z\rangle} \approx \mathrm{e}^{-1/4}\boldsymbol{x}. \tag{6.3}$$

It seems that the formal resemblance of the formula (6.3) and (2.19) referring to the case with the circular motion is not an accidental one. The range of application of (6.3) is the same as for (6.2), i.e. $|\boldsymbol{l}| \geqslant 10$. Because of the term $\mathrm{e}^{-1/4}$, it appears that the average value of $\boldsymbol{X}$ does not belong to the sphere with radius r. Proceeding analogously as in the case of the circle we introduce the relative average value of $\boldsymbol{X}$ of the form

$$\langle\langle X_i\rangle\rangle_z = \frac{\langle X_i\rangle_z}{\langle X_i\rangle_{w_i}} \qquad i = 1, 2, 3 \tag{6.4}$$

where $|\boldsymbol{w}_i\rangle$ is a coherent state with

$$\boldsymbol{w}_k = \cosh|\boldsymbol{l}|\boldsymbol{n}_k + \mathrm{i}\frac{\sinh|\boldsymbol{l}|}{|\boldsymbol{l}|}\boldsymbol{l}\times\boldsymbol{n}_k \qquad k = 1, 2, 3 \tag{6.5}$$

where $\boldsymbol{n}_k$ is the unit vector along the k coordinate axis and $\boldsymbol{l}$ is the same as in (6.1). In view of (6.3) and (6.4) we have

$$\langle\langle \boldsymbol{X}\rangle\rangle_z \approx \boldsymbol{x}. \tag{6.6}$$

Therefore, the relative expectation value $\langle\langle \boldsymbol{X}\rangle\rangle_z$ seems to be the most natural one to describe the average position of a particle on a sphere.

We have thus shown that the parameter $\boldsymbol{x}$ can be immediately related to the classical radius vector of a particle on a sphere. As with the case of circular motion (see formulae (2.18) and (2.21)), we interpret the relations (6.2) and (6.6) as the best possible approximation of the classical phase space. In this sense the coherent states labelled by points of such a deformed phase space are closest to the classical ones. The quantum fluctuations which are the reason for the approximate nature of (6.2) and (6.6) are in our opinion a characteristic feature of quantum mechanics on a sphere.

We finally remark that the discussion of the Heisenberg uncertainty relations analogous to that referring to the circle (see section 2) can also be performed in the case with the coherent states for a particle on a sphere. For example, a counterpart of the formula (2.22) is

$$(\Delta\boldsymbol{J})^2 \geqslant \frac{1}{2}\frac{\frac{1}{2}\,\mathrm{Tr}\langle V\rangle^2}{1-\frac{1}{2}\,\mathrm{Tr}\langle V\rangle^2} \tag{6.7}$$

where according to equation (4.2) we have $\langle V\rangle = \frac{1}{r}\boldsymbol{\sigma}\cdot\langle\boldsymbol{X}\rangle$. Such a discussion as well as the detailed analysis of the Heisenberg uncertainty relations for the quantum mechanics on a compact manifold will be the subject of a separate paper which is in preparation.

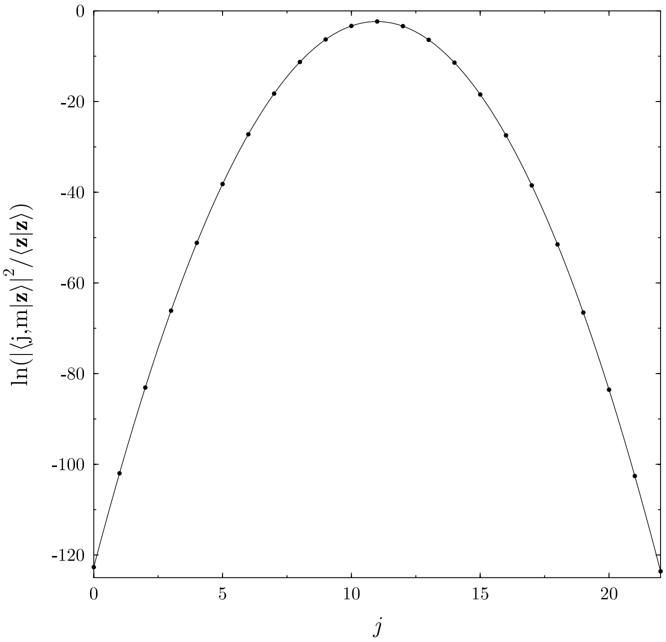

Figure 1. The plot of $\ln p_{j,m}$ (see (7.4)) with fixed $m = 0$ and z given by (6.1) where $\boldsymbol{x} = (0.412, 0.412, 0.812)$ and $\boldsymbol{l} = (8.124, -8.124, 0)$. Since $\boldsymbol{l}^2 = 132$, therefore $j_{\max} = 11$ coincides with the positive root of equation (7.5).

7. Simple application: the rotator

We now illustrate the actual treatment by the example of a free twist-0 particle on a sphere, i.e. the rotator. The corresponding Hamiltonian is given by

$$\hat{H} = \tfrac{1}{2}\boldsymbol{J}^2. \tag{7.1}$$

By (3.7*a*) the normalized solution of the Schrödinger equation

$$\hat{H}|E\rangle = E|E\rangle \tag{7.2}$$

can be expressed by

$$|E\rangle = |j, m; r\rangle \qquad E = \tfrac{1}{2}j(j+1). \tag{7.3}$$

We now discuss the distribution of the energies in the coherent state. The computer simulations indicate that the function

$$p_{j,m}(\boldsymbol{x}, \boldsymbol{l}) = \frac{|\langle j, m; r|z\rangle|^2}{\langle z|z\rangle} \qquad -j \leqslant m \leqslant j \tag{7.4}$$

determined by (5.14) and (6.1), which gives the probability of finding the system in the state $|j, m; r\rangle$, when the system is in the normalized coherent state $|z\rangle/\sqrt{\langle z|z\rangle}$, has the following properties. For fixed integer $m = l_3$ the function $p_{j,m}$ has a maximum at $j_{\max}$ coinciding with the integer nearest to the positive root of the equation

$$j(j+1) = \boldsymbol{l}^2 \tag{7.5}$$

(see figure 1). Thus, it turns out that the parameter $\frac{1}{2}\boldsymbol{l}^2$ can be regarded as the energy of the particle. Further, for fixed integer j in $p_{j,m}(\boldsymbol{x}, \boldsymbol{l})$ (see figure 2), such that (7.5) holds, the

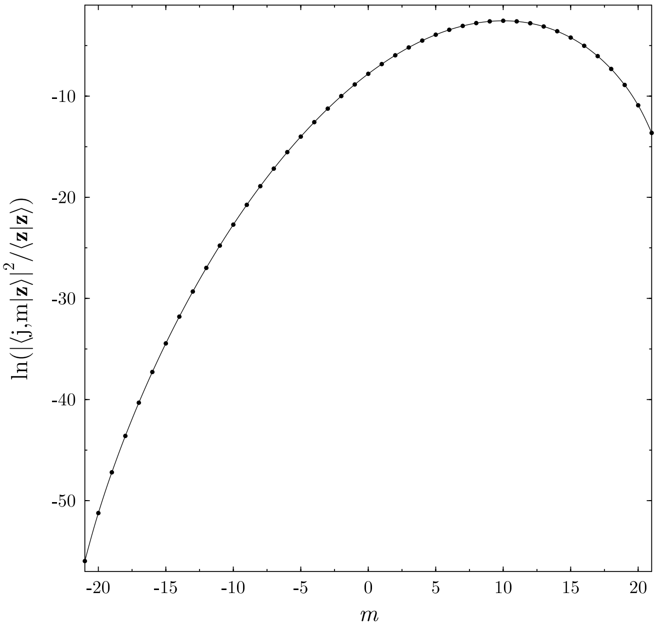

Figure 2. The plot of ln $p_{j,m}$ with $\boldsymbol{x} = (0.411, 0.911, 0.036)$ and $\boldsymbol{l} = (-17.490, 7.490, 10)$. The fixed $j = 21$ corresponds to the positive root of (7.5) where $\boldsymbol{l}^2 = 462$. The function has the maximum at $m_{\max} = l_3 = 10$.

function $p_{j,m}$ has a maximum at $m_{\max}$ coinciding with the integer nearest to l_3. It thus appears that the parameter l_3 can be identified with the projection of the momentum on the x_3-axis.

8. Conclusion

In this work we have introduced coherent states for a quantum particle on a sphere. An advantage of the formalism used is that the coherent states are labelled by points of the classical phase space. The authors have not found alternative constructions of coherent states for a quantum mechanics on a sphere preserving this fundamental property of coherent states. As pointed out in section 6, the quantum fluctuations arising in the case of motion on a sphere are bigger than those which take place for circular motion. This observation is consistent with the appearance of the additional degree of freedom for motion on a sphere. We remark that as with the particle on a circle, we deal within the actual treatment with the deformation of the classical phase space expressed by the approximate relations (6.2) and (6.6). We also point out that besides (6.2) and (6.6) the quasi-classical character of the introduced coherent states is confirmed by the behaviour of the distribution of the energies investigated in section 7. It seems that the approach introduced in this paper is not restricted to the study of the quasi-classical aspects of the quantum motion on a sphere. For example, the results of this work would be of importance in the theory of quantum chaos. In fact, in this theory the kicked rotator is one of the most popular model systems. Because of the well known difficulties in the analysis of the Heisenberg uncertainty relations occurring in the case with observables having a compact spectrum, such as the position operator $\boldsymbol{X}$ satisfying the $e(3)$ algebra (3.4), we have not studied them herein. The analysis of the Heisenberg uncertainty relations as well as the discussion of the case with a non-vanishing twist will be performed in future work.

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